

Renormalizable $1/N_f$ expansion for field theories in extra dimensions

Dmitri I. Kazakov^{ab} and Grigory S.Vartanov^{ac}

^a*Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Russia*

^b*Institute for Theoretical and Experimental Physics, Moscow, Russia*

^c*University Center, Joint Institute for Nuclear Research, Dubna, Russia*

E-mail: kazakovd@thsun1.jinr.ru, vartanov@theor.jinr.ru

ABSTRACT: We demonstrate how one can construct renormalizable perturbative expansion in formally nonrenormalizable higher dimensional field theories. It is based on $1/N_f$ -expansion and results in a logarithmically divergent perturbation theory in arbitrary high space-time dimension. First, we consider a simple example of N -component scalar field theory and then extend this approach to Abelian and non-Abelian gauge theories with N_f fermions. In the latter case, due to self-interaction of non-Abelian fields the proposed recipe requires some modification which, however, does not change the main results. The resulting effective coupling is dimensionless and is running in accordance with the usual RG equations. The corresponding beta function is calculated in the leading order and is nonpolynomial in effective coupling. It exhibits either UV asymptotically free or IR free behaviour depending on the dimension of space-time. The original dimensionful coupling plays a role of a mass and is also logarithmically renormalized. We analyze also the analytical properties of a resulting theory and demonstrate that in general it acquires several ghost states with negative and/or complex masses. In the former case, the ghost state can be removed by a proper choice of the coupling. As for the states with complex conjugated masses, their contribution to physical amplitudes cancels so that the theory appears to be unitary.

KEYWORDS: Field Theories in Higher Dimensions, Renormalization Group.

Contents

1. Introduction	1
2. $1/N$ expansion. Scalar theory	3
3. Properties of the $1/N$ expansion	6
4. $1/N_f$ expansion. QED	8
5. $1/N_f$ expansion. QCD	10
6. Renormalization group in $1/N$ expansion	17
7. Analytical properties and unitarity	20
8. Conclusion	25

1. Introduction

Nowadays it is popular to consider theories in extra dimensions as possible candidates for models of physics beyond the Standard Model [1, 2]. Within the braneworld scenario one assumes that the matter fields are localized at the brane while the force carriers can travel in the bulk [3]. Sometimes other fields might also live in extra dimensions. This means that one has higher dimensional QFT at least at short distances. However, it can hardly be considered as a consistent quantum theory beyond the tree level because of a lack of renormalizable perturbative expansion. Indeed, the usual coupling has a negative dimension, thus leading to power increasing divergencies which are out of control. In our previous work [4], we studied the UV divergencies in scalar theories in extra dimensions within the perturbative expansion and demonstrated that although the leading divergences are governed by the one-loop diagrams even in the nonrenormalizable case, as was argued in [5], this does not help to conquer them.

Popular reasoning when dealing with extra dimensional theories relies on higher energy (string) theory which is supposed to cure all the UV problems while the low energy one is treated as an effective theory basically at the tree level. One way to do it is the Kaluza-Klein approach [6]. In this case, one takes the Fourier transform over the extra dimensions and obtains an infinite tower of states with quantized masses. Then one has to sum over all these states. This sum is usually divergent and a special prescription is needed to regularize it. Doubtfully, however, that this approach solves the problem of nonrenormalizability in extra dimensional theories. As was shown in [7], the properly renormalized four-dimensional

theory never forgets its higher dimensional origin. It has an explicit cut-off dependence and can only be treated as an effective theory [8].

Here we make an attempt to construct renormalizable expansion in such formally nonrenormalizable theories using the well known technique of the $1/N$ expansion [9, 10], where in the scalar case N is the number of the scalar field components and in gauge theories it is the number of fermion flavours N_f . The number of colours N_c is kept fixed. This approach was successfully applied to non-linear sigma-models in 3 dimensions [11] and to quantum gravity in four dimensions [12] both of which are nonrenormalizable by power counting. Effectively, as we will show below, it leads to higher derivative theories and causes the usual problems of unitarity, locality and causality. However, these problems could be overcome though the analysis was performed mainly in the leading order [11, 12].

We follow the approach of [11, 12] and apply it to theories in extra dimensions with the aim to construct renormalizable and unitary $1/N$ expansion suitable for perturbative calculations. We first consider scalar higher dimensional theories as an example [13] and then treat the gauge theories with fermions in the same way [14]. The resulting perturbation theory is shown to be renormalizable, logarithmically divergent in any dimension D and obtains an effective dimensionless expansion parameter. It is nonpolynomial in effective coupling, but polynomial in $1/N$ and obeys the usual properties of renormalizable theory. It might be either UV asymptotically free or IR free depending on the space-time dimension D . The original dimensionful coupling does not serve as an expansion parameter anymore and plays the role of mass which is also logarithmically divergent and multiplicatively renormalized.

Within the dimensional regularization technique [15] we performed the renormalization procedure in scalar and gauge theories in arbitrary odd space-time dimension and calculated a few terms of the $1/N$ expansion. Even dimensions, in principle, can also be treated by this method; however, they lead to some complications due to the appearance of log terms.

It is well known that the main problem of the $1/N$ expansion is to prove unitarity of a resulting theory since the analytical properties of the effective propagator change. When summing up the vacuum polarization diagrams to the denominator of a singlet field one gets an imaginary part and, in general, additional poles in the complex plane. These poles correspond to ghost states with the wrong metric and negative or complex masses. It is a common problem in any realization of the $1/N$ expansion [12, 16, 17]. Effectively, it leads to higher derivative terms which may result in dynamical instability [18]. This is another issue that we do not discuss here. Note, however, the existence of "benign" quantum mechanical higher-derivative systems, where the classical vacuum is stable with respect to small perturbations and the problems appear only at the nonperturbative level [19]. The question of unitarity in higher derivative gravity in four dimensions was discussed in [12], where the role of ghost states was emphasized. It was also shown [20] that the higher derivative operators do not always improve the UV behaviour due to subtleties in analytical continuation from Minkowski to Euclidean metric.

Below we consider the unitarity problem in detail and suggest a possible solution which seems to lead to a unitary theory in physical subspace.

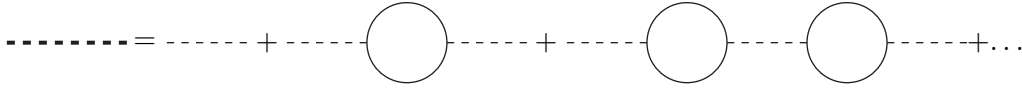


Figure 1: The chain of diagrams giving a contribution to the σ field propagator in the zeroth order of $1/N$ expansion

2. $1/N$ expansion. Scalar theory

To illustrate the method, we start with the scalar theory. Let us take the usual N component scalar field theory in D dimensions, where D takes an arbitrary value (> 4), with the ϕ^4 self-interaction. The Lagrangian looks like

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi})^2 - \frac{1}{2}m^2 \vec{\phi}^2 - \frac{\lambda}{8N}(\vec{\phi}^2)^2, \quad (2.1)$$

where N is the number of components of ϕ . We put N into the normalization of the coupling so that λ is fixed while $N \rightarrow \infty$. The theory is nonrenormalizable by power counting, the coupling λ has negative dimension $[\lambda] = 2 - D/2$. It is useful to rewrite eq. (2.1) introducing an auxiliary field σ [11]

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi})^2 - \frac{1}{2}m^2 \vec{\phi}^2 - \frac{1}{2\sqrt{N}}\sigma(\vec{\phi}^2) + \frac{1}{2\lambda}\sigma^2. \quad (2.2)$$

Now one has two fields, one N component and one singlet with triple interaction. Let us look at the propagator of the σ field. At the tree level it is just " $i\lambda$ ", but then one has to take into account the corrections due to the loops of ϕ (see figure 1).

If one follows the N dependence of the corresponding graphs, one finds out that it cancels: they are all of the zeroth order in $1/N$. Thus, one can sum them up and get

$$- - - - = - - - - \left(\frac{1}{1 - O - - -} \right) = \frac{i}{1/\lambda - \Pi(p^2)}, \quad (2.3)$$

where the polarization operator $\Pi(p^2)$ depends on D . In the massless case it looks like

$$\Pi(p^2) = -f(D)(-p^2)^{D/2-2}, \quad f(D) = \frac{\Gamma^2(D/2 - 1)\Gamma(2 - D/2)}{2^{D+1}\Gamma(D - 2)\pi^{D/2}}. \quad (2.4)$$

The obtained propagator (2.3) has a typical for $1/N$ expansion behaviour. Namely, it has a cut starting from $p^2 = 0$ (for $m = 0$, otherwise from $4m^2$) and poles at negative or complex p^2 depending on D . Notice that $f(D)$ is finite for any odd D , despite naive power counting, and diverges for even D . This is due to the use of dimensional regularization: the one-loop diagrams in odd dimensions are finite since the gamma function has poles only at integer negative arguments and not at half-integer ones. This phenomenon can also be understood in using other regularization techniques. In general one has the UV divergence which has to be subtracted. This subtraction requires the redefinition of simple

loop diagrams in D dimensions. However, the number of these diagrams is limited by $[D/2] - 1$, i.e., in 4 and 5 dimensions one has to define 1 diagram, in 6 and 7 dimensions - 2 diagrams, etc. In what follows we assume that this definition is made a'la dimensional regularization. Moreover, for simplicity of integration we limit ourselves to odd dimensions, which allows us to avoid the appearance of the log terms.

A special issue is the existence of poles in the propagator. Usually they signal of the appearance of new asymptotic states which raise the problem of unitarity of a resulting theory. We address this problem in more detail in section 7. Here we just mention that one can avoid poles on the real axis and have only complex conjugated pairs. This is enough for integration in Feynman diagrams.

Thus, we have now the modified Feynman rules: the ϕ propagator is the usual one while the σ propagator is given by eq. (2.3). One can then construct the diagrams using these propagators and the triple vertex having in mind that any closed cycle of ϕ gives an additional factor of N and any vertex gives $1/\sqrt{N}$.

Let us first analyse the degree of divergence. Let us start with the ϕ propagator. If the diagram with two external ϕ lines contains L loops, then it has $2L$ vertices, $2L - 1$ ϕ lines and L σ lines. Since each σ line now behaves like $1/p^{D-4}$, the degree of divergence is

$$\omega(G) = LD - (2L - 1)2 - L(D - 4) = 2! \tag{2.5}$$

for any D . Since this is a propagator, the divergence is proportional to p^2 and thus is reduced to the logarithmic one.

Let us now take the triple vertex. If it has L loops, then one has $2L + 1$ vertices, $2L$ ϕ lines and L σ lines. Hence, the degree of divergence is

$$\omega(G) = LD - (2L)2 - L(D - 4) = 0! \tag{2.6}$$

for any D . Thus, we again have only logarithmic divergence.

At last, consider the σ propagator. In L loops it has $2L$ vertices, $2L$ ϕ lines and $L - 1$ σ lines. The degree of divergence is

$$\omega(G) = LD - (2L)2 - (L - 1)(D - 4) = D - 4. \tag{2.7}$$

This means that in odd D it has no global divergence (again we explore the properties of dimensional regularization) and the only possible divergencies are those of the subgraphs eliminated by renormalization of ϕ and the coupling. To see this, consider a genuine diagram for the σ -field propagator which is shown in figure 2, where the blobs denote the 1PI vertex or propagator subgraphs.

After the R' operation¹ we do not have any poles in the integrand for the remaining one-loop integral. What is left is the finite part containing logarithms of momenta. This final integration has the following form:

$$\int \frac{\ln^n(k^2/\mu^2) \ln^m(k^2/p^2) \ln^k(k^2/(k-p)^2)}{k^2(k-p)^2} d^D k,$$

¹The R' operation means that we subtract from the diagram all divergent subgraphs

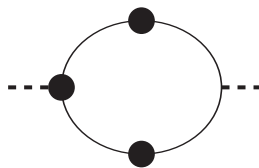


Figure 2: General type of the σ -field propagator

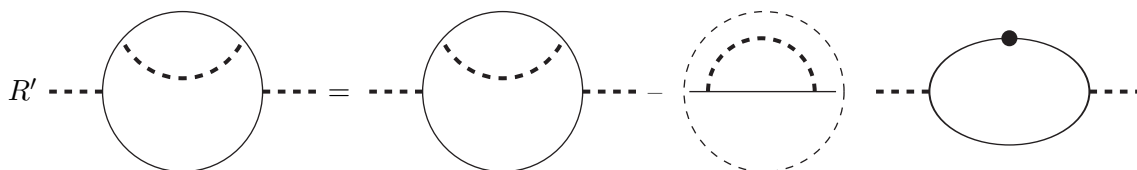


Figure 3: Demonstration of the global divergence cancellation in the two-loop diagram.

where n, m, k are some numbers. We ignore here all the masses since they do not contribute to the UV behaviour. Due to the naive power counting of divergences in dimensional regularization we obtain the result proportional to $\Gamma(2 - D/2)$ which is finite for any odd D . The logarithms can not change this property.

To demonstrate how this works explicitly, we consider a particular example of the two-loop diagram. The result of the R' -operation is shown in figure 3. After subtracting the divergence in a subgraph we have prior to the last integration

$$\int \frac{d^{D-2\epsilon}k}{k^2(p-k)^2} \left[\Gamma(-1 + \epsilon) \frac{\Gamma(D/2 - 1 - \epsilon)\Gamma(2 - \epsilon)}{\Gamma(D/2 - 2)\Gamma(D/2 + 1 - 2\epsilon)} \frac{1}{(k^2)^\epsilon} + \frac{1}{\epsilon} \frac{\Gamma(D/2 - 1)}{\Gamma(D/2 - 2)\Gamma(D/2 + 1)} \right].$$

The pole terms in the integrand cancel and expanding it over ϵ one gets $\log(k^2)$. It is, however, easier to integrate it without expanding over ϵ which gives

$$\Gamma(-1 + \epsilon) \frac{\Gamma^2(D/2 - 1 - \epsilon)\Gamma(2 - \epsilon)\Gamma(D/2 - 1 - 2\epsilon)\Gamma(2 + 2\epsilon - D/2)}{\Gamma(D/2 - 2)\Gamma(D/2 + 1 - 2\epsilon)\Gamma(1 + \epsilon)\Gamma(D - 2 - 3\epsilon)} \frac{(p^2)^{D/2-2}}{(p^2)^{2\epsilon}} + \frac{1}{\epsilon} \frac{\Gamma(D/2 - 1)}{\Gamma(D/2 - 2)\Gamma(D/2 + 1)} \frac{\Gamma^2(D/2 - 1 - \epsilon)\Gamma(2 + \epsilon - D/2)}{\Gamma(D - 2 - 2\epsilon)} \frac{(p^2)^{D/2-2}}{(p^2)^\epsilon} = O(1).$$

Thus, after the R' operation the diagram is finite and we do not need the σ field renormalization.

This way one gets the perturbative expansion with only logarithmic divergences. This is not expansion over dimensionful coupling λ but rather $1/N$ expansion with dimensionless parameter.

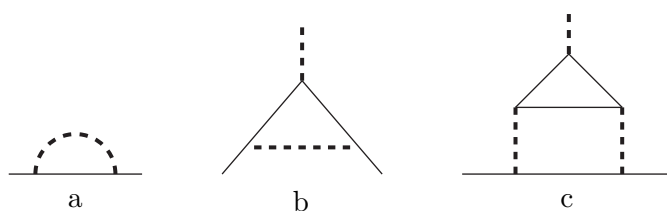


Figure 4: The leading order diagrams giving a contribution to the ϕ field propagator and the triple vertex in $1/N$ expansion

3. Properties of the $1/N$ expansion

Consider now the leading order calculations. We start with the $1/N$ terms for the propagator of ϕ and the triple vertex. One has the diagrams shown in figure 4. Notice that besides the one-loop diagrams in the same order of the $1/N$ expansion one has the two-loop diagram for the vertex.

Let us start with the diagram a). One has

$$I_a \sim \int \frac{d^{D'}k}{(2\pi)^{D'}N} \frac{1}{[(k-p)^2 - m^2][1/\lambda - \Pi(k^2)]}, \quad D' = D - 2\varepsilon.$$

Since we are interested in the UV behaviour we can omit the mass from the ϕ field propagator and "1/ λ " from the σ field propagator and take the massless limit of the polarization operator $\Pi(k^2)$. We will restore them when discussing the analytical properties. Then the UV asymptotics is given by

$$I_a \Rightarrow \int \frac{d^{D'}k}{(2\pi)^{D'}N f(D)} \frac{1}{(k-p)^2 (-k^2)^{D/2-2}}.$$

One can see that the original coupling λ plays the role of inverse mass and drops out from the UV expression. What is left is a dimensionless $1/N$ term.

Calculating the singular parts of the diagrams of figure 4 in dimensional regularization with $D' = D - 2\varepsilon$ one finds

$$\begin{aligned}
 \text{Diag. a} &\Rightarrow \frac{1}{\varepsilon N} A, & \text{Diag. b} &\Rightarrow \frac{1}{\varepsilon N} B, & \text{Diag. c} &\Rightarrow \frac{1}{\varepsilon N} C, & (3.1) \\
 A &= \frac{2\Gamma(D-2)}{\Gamma(D/2-2)\Gamma(D/2-1)\Gamma(D/2+1)\Gamma(2-D/2)}, \\
 B &= \frac{D}{4-D} A, \\
 C &= \frac{D(D-3)}{4-D} A.
 \end{aligned}$$

The corresponding renormalization constants in the \overline{MS} scheme then are

$$Z_2^{-1} = 1 - \frac{1}{\varepsilon} \frac{A}{N}, \quad (3.2)$$

$$Z_1 = 1 - \frac{1}{\varepsilon} \frac{B+C}{N}. \quad (3.3)$$

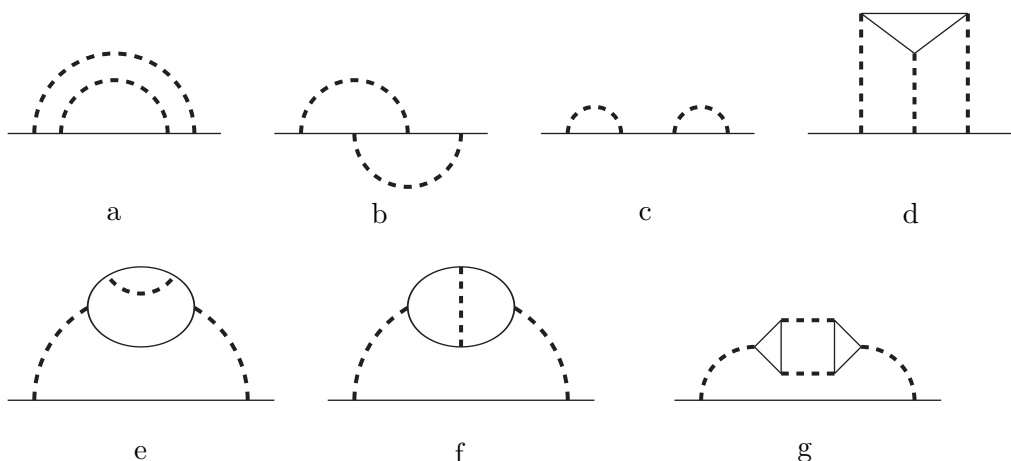


Figure 5: The second order diagrams giving a contribution to the ϕ field propagator in $1/N$ expansion

There is no any coupling in these formulas, its role is played by $1/N$ which is therefore infinitely renormalized. This seems to be unsatisfactory and to overcome this problem we introduce a new dimensionless coupling h associated with the triple vertex (and not with the σ propagator) as

$$\mathcal{L}_{int} = -\frac{\sqrt{h}}{2\sqrt{N}}\sigma\vec{\phi}^2.$$

Then in the leading order in $1/N$ the renormalization constants and the coupling take the form

$$Z_2^{-1} = 1 - \frac{h A}{\varepsilon N}, \tag{3.4}$$

$$Z_1 = 1 - \frac{h B}{\varepsilon N} - \frac{h^2 C}{\varepsilon N}, \tag{3.5}$$

$$h_B = (\mu^2)^\varepsilon h Z_1^2 Z_2^{-2} = h \left(1 - \frac{h 2(A+B)}{\varepsilon N} - \frac{h^2 C}{\varepsilon N} \right). \tag{3.6}$$

This is not, however, the final expression. To see this, we consider the next order of the $1/N$ expansion. The corresponding diagrams for the ϕ propagator are shown in figure 5. Again one can see that the $1/N^2$ terms contain not only the two-loop diagrams but also the three- and even four-loop ones.

All these diagrams are double logarithmically divergent, i.e., contain both single and double poles in dimensional regularization. We calculate the leading double pole after subtraction of the divergent subgraphs, i.e. perform the R' -operation. The answer is:

$$\begin{aligned} \text{Diag.a} &\Rightarrow -\frac{1}{\varepsilon^2 N^2} \frac{1}{2} A^2 h^2, & \text{Diag.b} &\Rightarrow -\frac{1}{\varepsilon^2 N^2} A B h^2, & \text{Diag.c} &\Rightarrow -\frac{1}{\varepsilon^2 N^2} A^2, \\ \text{Diag.d} &\Rightarrow -\frac{1}{\varepsilon^2 N^2} \frac{4}{3} A C h^3, & \text{Diag.e} &\Rightarrow -\frac{1}{\varepsilon^2 N^2} \frac{2}{3} A^2 h^3, & \text{Diag.f} &\Rightarrow -\frac{1}{\varepsilon^2 N^2} \frac{2}{3} A B h^3, \\ \text{Diag.g} &\Rightarrow -\frac{1}{\varepsilon^2 N^2} A C h^4. \end{aligned} \tag{3.7}$$

Here we face a problem, namely, subtracting the divergent subgraphs in the graphs e-g, we get the diagram which is absent in our expansion, since it is already included in our bold σ line (see figure 6).

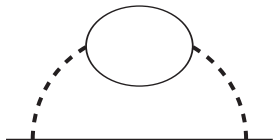


Figure 6: The "forbidden" loop diagram

There would be no problem unless this diagram is needed to match the so-called pole equations [21] which allow one to calculate the higher order poles in the Z factors from the single one. However, if we include this diagram in the σ line, it will not change the latter, except for the additional h factor coming from the vertex and not compensated by the propagator. Apparently, one can continue this insertion procedure

and add any number of such loops not changing the order of $1/N$ expansion. The result is the sum of a geometrical progression

$$\frac{1}{1+h},$$

which should multiply every σ line. Altogether this leads to the following effective Lagrangian for UV $1/N$ perturbation theory

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}(\partial_\mu \vec{\phi})^2 - \frac{\sqrt{h}}{2\sqrt{N}}\sigma(\vec{\phi}^2) + \frac{1}{2\lambda}\sigma^2 + \frac{1}{2}f(D)\sigma(\partial^2)^{D/2-2}\sigma(1+h). \quad (3.8)$$

Having all this in mind we come to the final expressions for the Z factors within the $1/N$ expansion:

$$Z_1 = 1 - \frac{1}{\varepsilon N} \left(\frac{Bh}{1+h} + \frac{Ch^2}{(1+h)^2} \right) + O\left(\frac{1}{N^2}\right), \quad (3.9)$$

$$Z_2^{-1} = 1 - \frac{1}{\varepsilon N} \frac{Ah}{1+h} + \frac{1}{\varepsilon^2 N^2} \left(\frac{3}{2} \frac{A^2 h^2}{(1+h)^2} + \frac{ABh^2}{(1+h)^2} + \frac{2}{3} \frac{A^2 h^3}{(1+h)^3} + \frac{2}{3} \frac{ABh^3}{(1+h)^3} + \frac{4}{3} \frac{ACH^3}{(1+h)^3} + \frac{ACH^4}{(1+h)^4} \right) + O\left(\frac{1}{\varepsilon N^2}\right). \quad (3.10)$$

4. $1/N_f$ expansion. QED

Let us consider now the usual QED with N_f fermion fields in D dimensions, where D takes an arbitrary odd value. The Lagrangian looks like

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2\alpha}(\partial_\mu A_\mu)^2 + i\bar{\psi}_i \hat{\partial} \psi_i - m\bar{\psi}_i \psi_i + \frac{e}{\sqrt{N_f}}\bar{\psi}_i \hat{A} \psi_i. \quad (4.1)$$

According to the general strategy, we now have to consider the photon propagator. Since due to the gauge invariance the polarization operator is transverse, it is useful to consider a transverse (Landau) gauge. This is not necessary but simplifies the calculations. Then in the leading order of the $1/N$ expansion one has the following sequence of bubbles (see figure 7) summed up into a geometrical progression. This is nothing more than the renormalon chain [22]. The resulting photon propagator takes the form similar to that for an auxiliary field σ in scalar case

$$D_{\mu\nu}(p) = -\frac{i}{p^2} \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \frac{1}{1 + e^2 f(D)(-p^2)^{D/2-2}}, \quad (4.2)$$



Figure 7: The chain of diagrams giving a contribution to the A field propagator in the zeroth order of the $1/N_f$ expansion

where

$$f(D) = \frac{\Gamma^2(D/2)\Gamma(2 - D/2)}{2^{D-[D/2]-1}\Gamma(D)\pi^{D/2}}$$

and we put $m = 0$ for simplicity.

This practically coincides with the expression obtained in scalar theory and all the following steps just repeat those in the latter. We change the normalization of the gauge field $A_\mu \rightarrow A_\mu/e$ and introduce the dimensionless coupling h associated with the triple vertex, so the effective Lagrangian takes the form

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & -\frac{1}{4}F_{\mu\nu} \left(\frac{1}{e^2} + f(D)(\partial^2)^{D/2-2}(1+h) \right) F_{\mu\nu} - \frac{1}{2\alpha e^2}(\partial_\mu A_\mu)^2 \\ & + i\bar{\psi}_i \hat{\partial} \psi_i - m\bar{\psi}_i \psi_i + \frac{\sqrt{h}}{\sqrt{N_f}} \bar{\psi}_i \hat{A} \psi_i. \end{aligned} \quad (4.3)$$

This new dimensionless coupling h enters into the gauge transformation and plays the role of a gauge charge. The old coupling e , on the contrary, is dimensionful and acts as a mass parameter in a gauge propagator. Since the coupling constant h is dimensionless the effective Lagrangian (4.3) when omitting the first term is conformal as considered in [23] where the theory was taken in $D = 3$.

Again, one has the modified Feynman rules with the photon propagator that decreases in the Euclidean region like $1/(p^2)^{D/2-1}$, thus improving the UV behaviour in a theory. The only divergent graphs are those of the fermion propagator and the triple vertex. They are both logarithmically divergent for any odd D . The photon propagator is genuinely finite and may contain divergencies only in subgraphs. One basically has the same graphs as in a scalar theory but with solid lines being the fermion ones and the dashed lines being the photon one.

The only difference (or simplification) comes from the Furry theorem and the gauge invariance. Namely, all triangles with three photon external lines vanish due to the Furry theorem and the gauge invariance which connects the fermion propagator with the triple vertex implies that $Z_1 = Z_2$. This relation holds in the $1/N_f$ expansion like in the usual PT. Thus, using the notation of a previous section, in the leading order one has

$$A = \frac{\Gamma(D)(D-1)(2-D/2)}{2^{[D/2]+1}\Gamma(2-D/2)\Gamma(D/2+1)\Gamma^2(D/2)}, \quad B = -A, \quad C = 0. \quad (4.4)$$

The same results were obtained in [24] where the author calculated the anomalous dimensions at the D -dimensional critical point where the fields obey asymptotic scaling and are

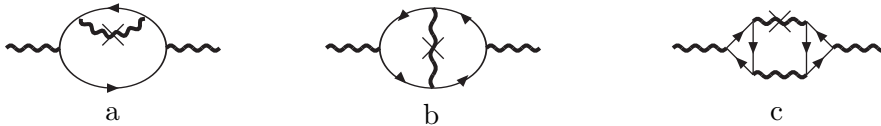


Figure 8: The first order diagrams giving a contribution to the $1/e^2$ renormalization in the $1/N_f$ expansion

conformal. This leads to the following renormalization constants in the leading order in $1/N_f$:

$$Z_1 = Z_2 = 1 + \frac{1}{\varepsilon N_f} \frac{Ah}{(1+h)}, \quad Z_3 = 1 \quad (4.5)$$

and, consequently, $h_B = h$. Hence, in odd-dimensional QED in the leading order of the $1/N_f$ expansion one does not need the coupling constant renormalization; only the wave function renormalization remains. This means that the coupling is not running.

In the second order one again has the same diagrams as in a scalar theory but with vanishing triangles. The renormalization constant in the second order is also essentially simplified compared to the scalar case and looks like

$$Z_1 = 1 + \frac{1}{\varepsilon N_f} \frac{Ah}{1+h} + \frac{1}{\varepsilon^2 N_f^2} \frac{1}{2} \frac{A^2 h^2}{(1+h)^2} + O\left(\frac{1}{\varepsilon N_f^2}\right). \quad (4.6)$$

Like in the scalar case the original dimensionful coupling e is not an expansion parameter anymore, but plays a role of a mass and is multiplicatively logarithmically renormalized. The leading order diagrams are shown in figure 8.

They give the following contribution:

$$\begin{aligned} \text{Diag.a} &\Rightarrow \frac{h^2}{\varepsilon N_f (1+h)^2} F, \quad \text{Diag.b} \Rightarrow \frac{h^2}{\varepsilon N_f (1+h)^2} E, \quad \text{Diag.c} \Rightarrow 0, \quad (4.7) \\ F &= \frac{\Gamma(D+1)(D/2-1)(D-1)^2(2-D/2)}{2^{D/2+1}\Gamma(2-D/2)\Gamma(D/2+2)\Gamma^2(D/2)}, \\ E &= -\frac{D^2 + D/2 - 9}{D/2(D/2-1)(D-1)} F. \end{aligned}$$

So one has

$$Z_{1/e^2} = 1 - \frac{1}{\varepsilon N_f} \left(\frac{(F+E)h^2}{(1+h)^2} \right) + O\left(\frac{1}{N_f^2}\right). \quad (4.8)$$

5. $1/N_f$ expansion. QCD

Consider now a non-Abelian theory with N_f fermions. Notice that in QCD, contrary to QED, all Feynman diagrams contain group factors so that the actual expansion parameter becomes N_c/N_f , thus requiring that this ratio is small. At the same time, to preserve

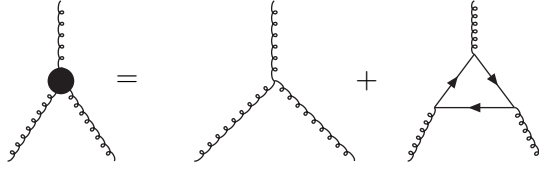


Figure 9: The diagrams giving a contribution to the A^3 term in the zeroth order of the $1/N_f$ expansion

asymptotic freedom in 4 dimensions one needs $N_c/N_f > 2/11$. So one has some interval where the N_c/N_f expansion might be valid. Of course, in non-Abelian theories the $1/N_c$ expansion would be preferable, since it accumulates the interactions of the gauge fields, however, in this case already the lowest approximation consists of all planar diagrams and is not known [27].

In the non-Abelian case one has some novel features due to the presence of the triple and quartic gauge vertices and the ghost fields. Similar to (4.1) we write down the Lagrangian for the gauge fields and N_f fermions as

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}^a)^2 - \frac{1}{2\alpha}(\partial_\mu A_\mu^a)^2 + i\bar{\psi}_i \hat{\partial} \psi_i - m\bar{\psi}_i \psi_i + \frac{g}{\sqrt{N_f}} \bar{\psi}_i \hat{A}^a T^a \psi_i + \partial_\mu \bar{c}^a D_\mu c_a,$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \frac{g}{\sqrt{N_f}} f^{abc} A_\mu^b A_\nu^c, \quad D_\mu = \partial_\mu + \frac{g}{\sqrt{N_f}} [A_\mu,]$$

Like in QED we choose the Landau gauge and sum up the fermion bubble diagrams into the denominator of the gauge field propagator

$$G_{\mu\nu}^{ab} = -\frac{i\delta^{ab}}{p^2} \frac{(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2})}{1 + g^2 f(D)(-p^2)^{D/2-2}}, \quad (5.1)$$

where the coefficient $f(D)$ differs from the Abelian case only by the colour factor $T(R)$

$$f(D) = \frac{\Gamma^2(D/2)\Gamma(2-D/2)}{2^{D-[D/2]-1}\Gamma(D)\pi^{D/2}} T(R)$$

and again we put $m = 0$ for simplicity.

In the non-Abelian case, contrary to the Abelian one, one has the triple and quartic self-interaction of the gauge fields. These vertices, which are suppressed by $1/\sqrt{N_f}$ and $1/N_f$, respectively, obtain loop corrections of the same order in $1/N_f$. The effective vertices in the leading order are given by the diagrams shown in figure 9 and 10.

Thus, besides the modification of the gauge propagator one has the modified vertices. The effective Lagrangian in the case of vertices is not given by a simple local expression due to complexity of the loop diagrams. So we keep it in the form of the diagrams which have to be evaluated in integer dimension. Due to the rules of dimensional regularization they

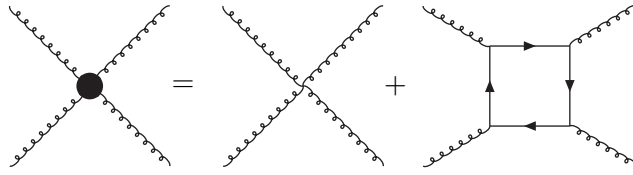


Figure 10: The diagrams giving a contribution to the A^4 term in the zeroth order of the $1/N_f$ expansion

are finite for any odd D , otherwise one has to redefine them. What is crucial, however, is that there are only three diagrams which have to be redefined. Hence, after rescaling the gauge field $A_\mu \rightarrow A_\mu/g$ one obtains the following effective Lagrangian:

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4g^2}(F_{\mu\nu}^a)^2 - \left(\text{fermion loop} + \text{ghost loop} + \text{ghost-fermion loop} \right) \quad (5.2)$$

$$- \frac{1}{2\alpha g^2}(\partial_\mu A_\mu^a)^2 + i\bar{\psi}_i \hat{\partial} \psi_i - m\bar{\psi}_i \psi_i + \frac{1}{\sqrt{N_f}} \bar{\psi}_i \hat{A}^a T^a \psi_i + \partial_\mu \bar{c}^a D_\mu c_a,$$

Notice that dimensionful coupling g drops from all terms except for the first one and is not an expansion parameter anymore.

Calculating the degree of divergence after summing up the diagrams of the zeroth order, similar to the scalar case and QED, one has only four types of logarithmically divergent diagrams: the fermion and the ghost propagators, the fermion-gauge-vertex and ghost-gauge-ghost vertex. The gauge propagator as well as pure gauge vertices are finite and may contain only divergent subgraphs.

The next step is the introduction of a dimensionless coupling h . Here one should be accurate since this coupling enters not only into the triple gauge-fermion vertex, but due to the gauge invariance should be present in gauge and gauge-ghost vertices. It should be the same in all three of them. In the case of a gauge theory, the coupling h enters the gauge transformation and acts as a gauge charge of the fermion and gauge fields.

When constructing the Feynman diagrams, one reproduces the one-loop cycles that are already present in the effective Lagrangian (5.2) but with additional factors h . In the scalar or QED case, this happened only for the propagator, but here it is also true for the vertices. As a result, the final expression for the effective Lagrangian takes the form

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4g^2}(F_{\mu\nu}^a)^2 - \left(\text{fermion loop} + \text{ghost loop} + \text{ghost-fermion loop} \right) (1+h) \quad (5.3)$$

$$- \frac{1}{2\alpha g^2}(\partial_\mu A_\mu^a)^2 + i\bar{\psi}_i \hat{\partial} \psi_i - m\bar{\psi}_i \psi_i + \frac{\sqrt{h}}{\sqrt{N_f}} \bar{\psi}_i \hat{A}^a T^a \psi_i + \partial_\mu \bar{c}^a D_\mu c_a,$$

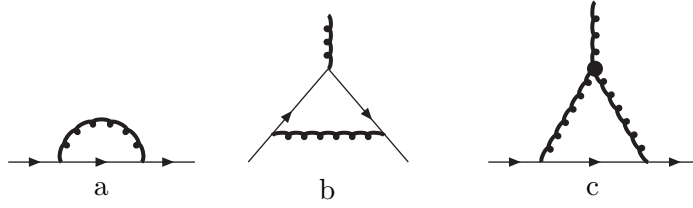


Figure 11: The leading order diagrams giving a contribution to the ψ field propagator and the triple vertex in $1/N_f$ expansion

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \frac{\sqrt{h}}{\sqrt{N_f}} f^{abc} A_\mu^b A_\nu^c, \quad D_\mu c_a = \partial_\mu c_a + \frac{\sqrt{h}}{\sqrt{N_f}} f^{abc} A_\mu^b c^c.$$

Consider now the leading order calculations. We start with the $1/N_f$ terms for the fermion and the triple fermion-gauge-fermion vertex. The diagrams are shown in figure 11. The first two are the same as in QED. The third diagram contains new effective vertex which includes the usual triple vertex and the fermion triangle. The usual vertex does not give a contribution since it is finite by a simple power counting. At the same time, the fermion triangle is momentum dependent and the resulting diagram is logarithmically divergent.

Calculating the singular parts of the diagrams of figure 11 in dimensional regularization with $D' = D - 2\varepsilon$ one finds

$$\begin{aligned} \text{Diag. a} &\Rightarrow \frac{1}{\varepsilon N_f} \frac{h}{1+h} A, & \text{Diag. b} &\Rightarrow \frac{1}{\varepsilon N_f} \frac{h}{1+h} B, & \text{Diag. c} &\Rightarrow \frac{1}{\varepsilon N_f} \frac{h}{1+h} C, \\ A &= \frac{\Gamma(D)(D-1)(2-D/2)C_F}{2^{[D/2]+1}\Gamma(2-D/2)\Gamma(D/2+1)\Gamma^2(D/2)T}, \\ B &= -\frac{C_F - C_A/2}{C_F} A, \\ C &= -\frac{(1-D/2)C_A}{2(2-D/2)C_F} A, \end{aligned} \tag{5.4}$$

which is again in agreement with [25]. Notice that the third diagram is proportional to $h/(1+h)$ instead of $h^2/(1+h)^2$ as in the scalar case. The reason is that now we have an effective triple gauge vertex proportional to $\sqrt{h}(1+h)$ instead of $\sqrt{h}h$ that cancels one factor of $h/(1+h)$.

Therefore, in the leading order in the $1/N_f$ expansion the renormalization constants take the form

$$Z_2^{-1} = 1 - \frac{1}{\varepsilon N_f} \frac{Ah}{(1+h)}, \tag{5.5}$$

$$Z_1 = 1 - \frac{1}{\varepsilon N_f} \frac{(B+C)h}{(1+h)}, \tag{5.6}$$

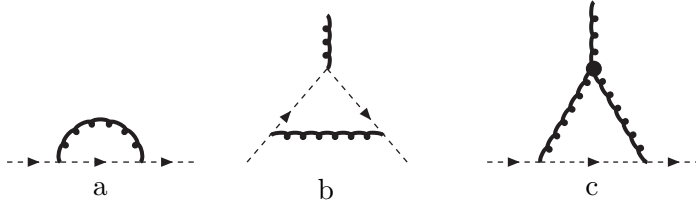


Figure 12: The leading order diagrams giving a contribution to the ghost field propagator and the triple vertex in $1/N_f$ expansion

$$Z_h = Z_1^2 Z_2^{-2} = 1 - \frac{1}{\varepsilon N_f} \frac{2(A+B+C)h}{(1+h)}. \quad (5.7)$$

To check the gauge invariance, we calculated the renormalization of the coupling through the gauge-ghost interaction. The leading diagrams are shown in figure 12.

Calculating the singular parts of the diagrams in dimensional regularization one finds

$$\begin{aligned} \text{Diag.a} &\Rightarrow \frac{1}{\varepsilon N_f} \frac{h}{1+h} A', & \text{Diag.b} &\Rightarrow \frac{1}{\varepsilon N_f} \frac{h}{1+h} B', & \text{Diag.c} &\Rightarrow \frac{1}{\varepsilon N_f} \frac{h}{1+h} C', \\ A' &= \frac{\Gamma(D)(D-1)C_A}{2^{[D/2]+2}\Gamma(2-D/2)\Gamma(D/2+1)\Gamma^2(D/2)T}, & (5.8) \\ B' &= 0, \\ C' &= 0, \end{aligned}$$

which gives the following renormalization constants in the ghost sector

$$\tilde{Z}_1 = 1, \quad (5.9)$$

$$\tilde{Z}_2^{-1} = 1 - \frac{1}{\varepsilon N_f} \frac{A'h}{1+h}, \quad (5.10)$$

$$Z_h = \tilde{Z}_1^2 \tilde{Z}_2^{-2} = 1 - \frac{2}{\varepsilon N_f} \frac{A'h}{1+h}. \quad (5.11)$$

One can see that the following relation holds:

$$A+B+C = A'+B'+C', \quad (5.12)$$

which follows from the gauge invariance.

We look now at the next-to-leading order to compare it with the scalar case. The corresponding diagrams for the fermion propagator are shown in figure 13. They require some explanation. The first line of diagrams in figure 13 is obtained from the one-loop diagrams of figure 11 by inserting into the vertex or the fermion line of the one-loop divergent subgraphs from figure 11. For example, the diagram *d* in figure 13 is the diagram *a* from figure 11 with divergent one-loop subgraph *c* from figure 11 substituted instead of the initial vertex. The second line of the diagrams in figure 13 is obtained from the "forbidden" diagram of figure 14 by inserting the same one-loop divergent subgraphs from

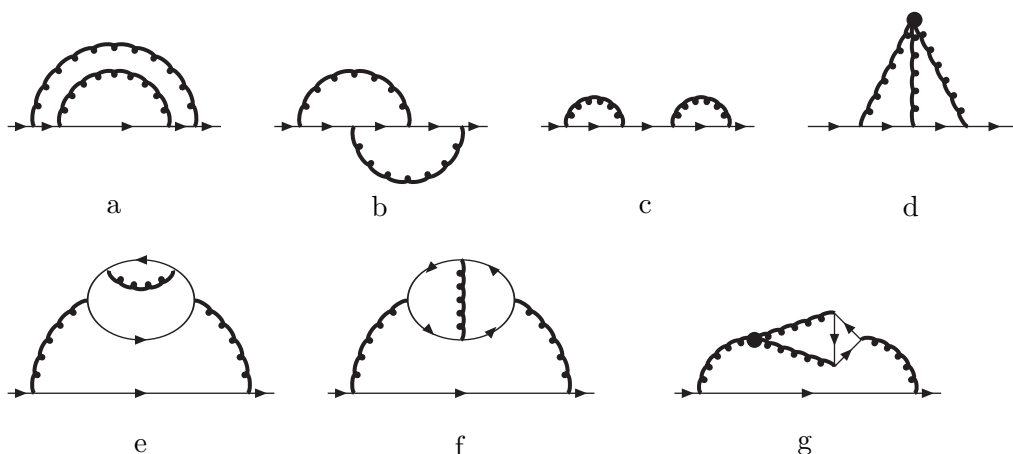


Figure 13: The second order diagrams giving a contribution to the fermion propagator in the $1/N_f$ expansion

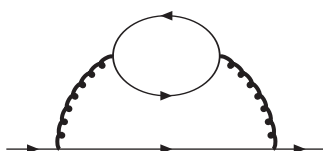


Figure 14: The "forbidden" loop diagram

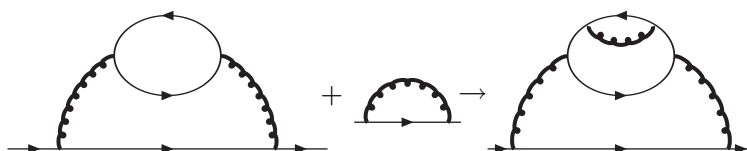


Figure 15: The diagram e from figure 13 as a result of insertion of the diagram a from figure 11 into the fermion line.

figure 11. The diagram e is the diagram of figure 14 with insertion of the subgraph a from figure 11 into the fermion line (see figure 15) and the diagram g comes from the insertion of the subgraph c from figure 11 instead of one of the vertices in the fermion loop (see figure 16).

All these diagrams are double logarithmically divergent, i.e., contain both single and double poles in dimensional regularization. We calculated the leading double poles after

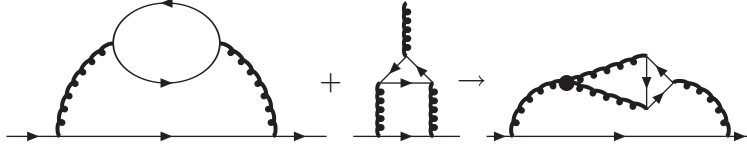


Figure 16: The diagram g from figure 13 as a result of insertion of the diagram c from figure 11 instead of one of the vertices in the diagram from figure 14.

subtraction of the divergent subgraphs, i.e., performed the R' -operation. The answer is:

$$\begin{aligned}
 \text{Diag.}a &\Rightarrow -\frac{1}{\varepsilon^2 N_f^2} \frac{A^2 h^2}{2(1+h)^2}, & \text{Diag.}b &\Rightarrow -\frac{1}{\varepsilon^2 N_f^2} \frac{ABh^2}{(1+h)^2}, & \text{Diag.}c &\Rightarrow -\frac{1}{\varepsilon^2 N_f^2} \frac{A^2 h^2}{(1+h)^2}, \\
 \text{Diag.}d &\Rightarrow -\frac{1}{\varepsilon^2 N_f^2} \frac{ACH^2}{(1+h)^2}, & \text{Diag.}e &\Rightarrow \frac{1}{\varepsilon^2 N_f^2} \frac{2A^2 h^3}{3(1+h)^3}, \\
 \text{Diag.}f &\Rightarrow \frac{1}{\varepsilon^2 N_f^2} \frac{2ABh^3}{3(1+h)^3}, & \text{Diag.}g &\Rightarrow \frac{1}{\varepsilon^2 N_f^2} \frac{2ACH^3}{3(1+h)^3}.
 \end{aligned} \tag{5.13}$$

We performed also the calculation for the fermion-gauge-fermion vertex but do not present the diagram-by-diagram result because of the lack of space and give only the final answer.

Having all this in mind we come to the final expressions for the Z factors in the second order of the $1/N_f$ expansion in the fermion sector:

$$\begin{aligned}
 Z_1 = 1 - \frac{1}{\varepsilon N_f} \frac{(B+C)h}{1+h} + \frac{1}{\varepsilon^2 N_f^2} \left(\frac{3}{2} \frac{(B+C)^2 h^2}{(1+h)^2} + \frac{A(B+C)h^2}{(1+h)^2} \right. \\
 \left. - \frac{2}{3} \frac{(B+C)^2 h^3}{(1+h)^3} - \frac{2}{3} \frac{A(B+C)h^3}{(1+h)^3} \right) + O\left(\frac{1}{\varepsilon N_f^2}\right),
 \end{aligned} \tag{5.14}$$

$$\begin{aligned}
 Z_2^{-1} = 1 - \frac{1}{\varepsilon N_f} \frac{Ah}{1+h} + \frac{1}{\varepsilon^2 N_f^2} \left(\frac{3}{2} \frac{A^2 h^2}{(1+h)^2} + \frac{A(B+C)h^2}{(1+h)^2} \right. \\
 \left. - \frac{2}{3} \frac{A(A+B+C)h^3}{(1+h)^3} \right) + O\left(\frac{1}{\varepsilon N_f^2}\right).
 \end{aligned} \tag{5.15}$$

The same calculation in the ghost sector gives

$$\begin{aligned}
 \tilde{Z}_1 &= 1, \\
 \tilde{Z}_2^{-1} &= 1 - \frac{1}{\varepsilon N_f} \frac{A'h}{1+h} + \frac{1}{\varepsilon^2 N_f^2} \left(\frac{3}{2} \frac{A'^2 h^2}{(1+h)^2} - \frac{2}{3} \frac{A'(A+B+C)h^3}{(1+h)^3} \right) + O\left(\frac{1}{\varepsilon N_f^2}\right).
 \end{aligned} \tag{5.16}$$

Notice the absence of the ghost-gauge-ghost vertex renormalization.

The final second order expression for the coupling renormalization calculated in both ways having in mind relation (5.12) is

$$Z_h = 1 - \frac{1}{\varepsilon N_f} \frac{2(A+B+C)h}{1+h} + \frac{1}{\varepsilon^2 N_f^2} \left(4 \frac{(A+B+C)^2 h^2}{(1+h)^2} - \frac{4}{3} \frac{(A+B+C)^2 h^3}{(1+h)^3} \right) + O\left(\frac{1}{\varepsilon N_f^2}\right). \tag{5.17}$$

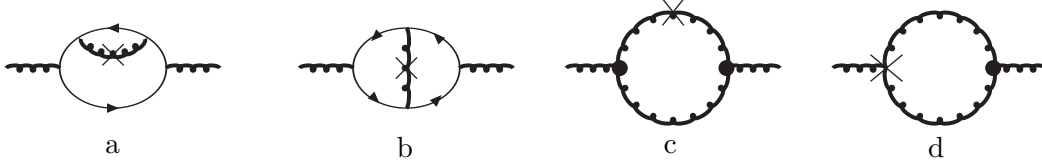


Figure 17: The first order diagrams giving a contribution to the $1/g^2$ renormalization in $1/N_f$ expansion

Like in the scalar and QED case, one can also calculate the renormalization of the original coupling g^2 . The leading order diagrams are shown in figure 17 which give the following singular parts like in [25]

$$\begin{aligned}
 \text{Diag.a} &\Rightarrow \frac{1}{\varepsilon N_f} \frac{h^2}{(1+h)^2} F, & \text{Diag.b} &\Rightarrow \frac{1}{\varepsilon N_f} \frac{h^2}{(1+h)^2} E, \\
 \text{Diag.c} &\Rightarrow \frac{1}{\varepsilon N_f} \frac{h^2}{(1+h)^2} G, & \text{Diag.d} &\Rightarrow \frac{1}{\varepsilon N_f} \frac{h^2}{(1+h)^2} H, \\
 F &= \frac{\Gamma(D+1)(D/2-1)(D-1)^2(2-D/2)C_F}{2^{[D/2]+2}\Gamma(2-D/2)\Gamma(D/2+2)\Gamma^2(D/2)T(R)}, \\
 E &= -\frac{D^2+D/2-9}{D/2(D/2-1)(D-1)} \frac{C_F-C_A/2}{C_F} F, \\
 G &= \frac{4(D/2)^6-6(D/2)^5+18(D/2)^4-67(D/2)^3+85(D/2)^2-19D+6}{2(D-1)^2(1-D/2)^2(2-D/2)D} \frac{C_A}{C_F} F, \\
 H &= \frac{D^3-D^2/2-2D+1}{D(1-D/2)(2-D/2)(D-1)^2} \frac{C_A}{C_F} F.
 \end{aligned} \tag{5.18}$$

The corresponding renormalization constant looks like

$$Z_{1/g^2} = 1 - \frac{1}{\varepsilon N_f} \frac{(F+E+G+H)h^2}{(1+h)^2}. \tag{5.19}$$

6. Renormalization group in $1/N$ expansion

Having these expressions for the Z factors one can construct the coupling constant renormalization and the corresponding RG functions. One has as usual in the dimensional regularization

$$h_B = (\mu^2)^\varepsilon h Z_1^2 Z_2^{-2} = (\mu^2)^\varepsilon \left(h + \sum_{n=1}^{\infty} \frac{a_n(h, N)}{\varepsilon^n} \right), \tag{6.1}$$

$$Z_i = 1 + \sum_{n=1}^{\infty} \frac{c_n^i(h, N)}{\varepsilon^n}, \tag{6.2}$$

where the first coefficients a_n and c_n^i can be deduced from the Z factors.

This allows one to get the anomalous dimensions and the beta function defined as

$$\gamma(h, N) = -\mu^2 \frac{d}{d\mu^2} \log Z = h \frac{d}{dh} c_1, \quad (6.3)$$

$$\beta(h, N) = 2h(\gamma_1 + \gamma_2) = \left(h \frac{d}{dh} - 1 \right) a_1. \quad (6.4)$$

We first consider the scalar case. With the help of eqs. (3.9), (3.10) one gets in the leading order of $1/N$ expansion²

$$\gamma_2(h, N) = -\frac{1}{N} \frac{Ah}{(1+h)^2}, \quad (6.5)$$

$$\gamma_1(h, N) = -\frac{1}{N} \left(\frac{Bh}{(1+h)^2} + \frac{2Ch^2}{(1+h)^3} \right),$$

$$\beta(h, N) = -\frac{1}{N} \left(\frac{2(A+B)h^2}{(1+h)^2} + \frac{4Ch^3}{(1+h)^3} \right). \quad (6.6)$$

It is instructive to check the so-called pole equations [21] that express the coefficients of the higher order poles in ε of the Z factors via the coefficients of a simple pole. For Z_2^{-1} one has, according to (3.10),

$$c_1(h, N) = -\frac{1}{N} \frac{Ah}{1+h}, \quad (6.7)$$

$$c_2(h, N) = \frac{1}{N^2} \left(\frac{3}{2} \frac{A^2 h^2}{(1+h)^2} + \frac{ABh^2}{(1+h)^2} + \frac{2}{3} \frac{A^2 h^3}{(1+h)^3} \right. \\ \left. + \frac{2}{3} \frac{ABh^3}{(1+h)^3} + \frac{4}{3} \frac{ACH^3}{(1+h)^3} + \frac{ACH^4}{(1+h)^4} \right). \quad (6.8)$$

At the same time the coefficient c_2 can be expressed through c_1 via the pole equations as

$$h \frac{dc_2}{dh} = \gamma_2 c_1 + \beta \frac{dc_1}{dh}, \quad (6.9)$$

which gives

$$h \frac{dc_2}{dh} = \frac{1}{N^2} \frac{Ah}{(1+h)^2} \frac{Ah}{1+h} + \frac{1}{N^2} \left(\frac{2(A+B)h^2}{(1+h)^2} + \frac{4Ch^3}{(1+h)^3} \right) \frac{A}{(1+h)^2}.$$

Integrating this equation one gets for c_2 the expression coinciding with (6.8) which was obtained by direct diagram evaluation. Notice that to get this coincidence the h -dependence in the denominator of eqs. (3.9), (3.10) was absolutely crucial.

We have also checked the pole equations for the renormalized coupling. They look as follows

$$\left(h \frac{d}{dh} - 1 \right) a_n = \beta \frac{da_{n-1}}{dh}. \quad (6.10)$$

In the leading order in h when

$$a_1(h, N) \simeq -\frac{2(A+B)h^2}{N} \quad \text{and} \quad \beta(h, N) \simeq -\frac{2(A+B)h^2}{N}$$

²Note that the anomalous dimension of a field γ_2 , is defined with respect to Z_2^{-1} .

one should have a geometric progression

$$a_n(h, N) = a_1(h, N)^n.$$

We have checked this relation up to three loops and confirmed its validity.

Having expression for the β function one may wonder how the coupling is running. The crucial point here is the sign of the β function. One has

$$\beta(h, N) = -\frac{1}{N} \frac{4\Gamma(D-2) \left(\frac{2h^2}{(1+h)^2} + \frac{D(D-3)h^3}{(1+h)^3} \right)}{\Gamma(D/2-2)\Gamma(D/2-1)\Gamma(D/2+1)\Gamma(3-D/2)}. \quad (6.11)$$

It can also be rewritten as

$$\beta(h, N) = -\frac{1}{N} \frac{2^{D-1}\Gamma(D/2-1/2)(-)^{(D-1)/2} \left(\frac{2h^2}{(1+h)^2} + \frac{D(D-3)h^3}{(1+h)^3} \right)}{\Gamma(1/2)\pi\Gamma(D/2+1)}, \quad (6.12)$$

that clearly indicates that the theory is UV asymptotically free for $D = 2k + 1$, k - even and IR free for k -odd. Solution of the RG equation looks somewhat complicated, but for the small coupling in the leading order it simply equals the usual leading log approximation

$$h(t, h) \simeq \frac{h}{1 - \beta_0 h \log(t)}, \quad \beta_0 = -\frac{1}{N} \frac{2^D \Gamma(D/2-1/2)(-)^{(D-1)/2}}{\Gamma(1/2)\pi\Gamma(D/2+1)}. \quad (6.13)$$

For example, for $D = 5, 7$ the beta function equals $\beta_0 = -256/15\pi^2 N$ and $2^{12}/105\pi^2 N$, respectively.

We now come to the gauge theories. With the help of eqs. (5.14), (5.15) one gets in the leading order of the $1/N_f$ expansion

$$\gamma_2(h, N_f) = -\frac{1}{N_f} \frac{Ah}{(1+h)^2}, \quad \gamma_1(h, N_f) = -\frac{1}{N_f} \frac{(B+C)h}{(1+h)^2}, \quad (6.14)$$

$$\tilde{\gamma}_2(h, N_f) = -\frac{1}{N_f} \frac{A'h}{(1+h)^2}, \quad \tilde{\gamma}_1(h, N_f) = O\left(\frac{1}{N_f^2}\right), \quad (6.15)$$

$$\beta(h, N_f) = -\frac{1}{N_f} \frac{2(A+B+C)h^2}{(1+h)^2}, \quad (6.16)$$

The situation is similar to that in scalar theory. Only the value of coefficients are different. This, however, does not influence the pole equations. They remain to be valid.

Equation (5.17) gives us the sign of the beta function. In the leading order one has

$$\frac{dh}{dt} = \beta(h) = -\frac{\Gamma(D)(D-1)C_A}{2^{[D/2]+2}\Gamma(2-D/2)\Gamma(D/2+1)\Gamma^2(D/2)N_f T} \frac{h^2}{(1+h)^2}, \quad (6.17)$$

which means that contrary to the scalar case (6.12) $\beta(h) > 0$ for $D = 5$, $\beta(h) < 0$ for $D = 7$ and then alternates with D as in the scalar case.

Solution to eq. (6.17) for small h is again reduced to the usual one. As for the original couplings, there is no simple solution either except for the QED case, where the coupling h is not running and solution of the RG equation for $1/e^2$ with fixed h is

$$\frac{1}{e^2} = \frac{1}{e_0^2} \left(\frac{p^2}{p_0^2} \right)^\gamma, \tag{6.18}$$

with the anomalous dimension

$$\gamma = \frac{\Gamma(D)(D-1)(D/2-2)(D-3)(D+2)(D-6)}{2^{[D/2]+2}\Gamma(D/2+2)\Gamma^2(D/2)\Gamma(2-D/2)N_f} \frac{h^2}{(1+h)^3}.$$

The sign of γ depends on D . For $D = 5, 7$ $\gamma > 0$, for $D = 9$ $\gamma < 0$ and then alternates with every odd D . eq. (6.18) reminds the power law behaviour of the initial coupling in extra dimensions within the Kaluza-Klein approach [26] though anomalous dimension γ is different.

7. Analytical properties and unitarity

Consider now the analytical properties of the propagator and related problem of unitarity. The problem is common to scalar and gauge theories so for simplicity we concentrate on the sigma field propagator (2.3). Besides the cut starting from $4m^2$ it has poles in the complex p^2 plane. Hence, knowing the analytical structure, one can write down the Källén-Lehmann representation [28].

Let us first consider the massless case (2.4)

$$D(p^2) = \frac{i}{1/\lambda + f(D)(-p^2)^{D/2-2}}. \tag{7.1}$$

Depending on a sign of $f(D)$ there are two possibilities: either one has a pole at real axis and (possibly) pairs of complex conjugated poles ($f(D) < 0$, $D=5,9,\dots$) or one has only pairs of complex conjugated poles ($f(D) > 0$, $D=7,11,\dots$) and all the rest appears at the second Riemann sheet. We consider the cases of $D = 5$ and $D = 7$ as the nearest options. One has, respectively,

$$D_5(p^2) = -\frac{2(256\pi)^2\lambda^2}{p^2 + (256\pi/\lambda)^2} + \frac{1}{\pi} \int_0^\infty \frac{dm^2}{p^2 - m^2} \frac{256\pi\lambda^2\sqrt{m^2}}{(256\pi)^2 + \lambda^2 m^2}, \tag{7.2}$$

and

$$D_7(p^2) = -\frac{2}{3} \frac{(\frac{8192\pi^2}{\lambda})^{2/3} e^{\pi i/3}}{p^2 + (\frac{8192\pi^2}{\lambda})^{2/3} e^{-2\pi i/3}} - \frac{2}{3} \frac{(\frac{8192\pi^2}{\lambda})^{2/3} e^{-\pi i/3}}{p^2 + (\frac{8192\pi^2}{\lambda})^{2/3} e^{2\pi i/3}} + \frac{1}{\pi} \int_0^\infty \frac{dm^2}{p^2 - m^2} \frac{8192\pi^2\lambda^2(m^2)^{3/2}}{\lambda^2(m^2)^3 + (8192\pi^2)^2}. \tag{7.3}$$

Notice that the continuous spectrum has a positive spectral density and corresponds to production of real pairs of ϕ fields (or pairs of fermions in the gauge case). These states are present in the original spectrum and cause no problem with unitarity. This analysis

was performed at the tree level in [11] and can be extended to any number of loops. One can show that all the cuts imposed on diagrams when applying Cutkosky rules [29] in any order of perturbation theory lead to the usual asymptotic states on mass shell and no new states appear.

The problem comes with the poles. One can see that the pole terms come with negative sign and, therefore, correspond to the ghost states [16]. For $D = 5$ one has only one pole at the positive real semiaxis while for $D = 7$ one has a pair of complex conjugated poles, as shown in figure 18.

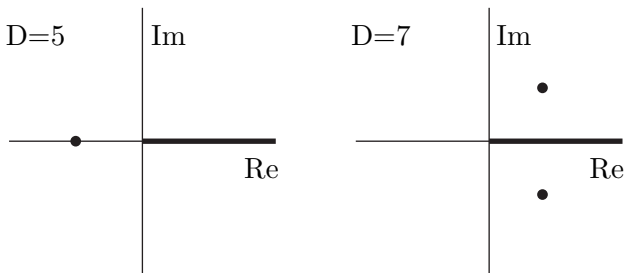


Figure 18: The analytical structure of the auxiliary field propagator

The presence of these ghost states is the drawback of a theory. They signal of instability of the vacuum state. Indeed, as it was shown in [30], the vacuum might be unstable with respect to appearance of condensates. This will lead to additional diagrams similar to those in QCD. However, they do not seem to improve the situation. Thus, one has either to try to get rid of ghost poles or to make sure that they do not give a contribu-

tion to physical amplitudes.

Let us first see what happens if one takes a nonzero mass of the ϕ field. The polarization operator then is

$$\Pi(p^2) = -\frac{\Gamma(2 - D/2)}{2^{D+1}\pi^{D/2}} \int_0^1 dx (-p^2 x(1-x) + m^2)^{D/2-2}. \quad (7.4)$$

For $D = 5$ one has

$$\Pi_5(p^2) = \frac{1}{32\pi^2} m \left[\frac{4\sqrt{a} + (a-4) \ln\left(\frac{2-\sqrt{a}}{2+\sqrt{a}}\right)}{8\sqrt{a}} \right], \quad (7.5)$$

where $a = \frac{p^2}{m^2}$. Since the existence of a pole is governed by the equation

$$\Pi(p^2) = 1/\lambda,$$

one has to check whether this equation is satisfied somewhere in the complex p^2 plane. Remind that for the massless case the pole exists at $p^2 = -(256\pi/\lambda)^2$. In figure 19, we show the plot of $\Pi_5(p^2)$ for real p^2 (left) and the absolute value in the complex plane (right).

One can see that for negative p^2 the polarization operator is always greater than 1 (in units of $\frac{1}{32\pi^2} m\lambda$), for positive p^2 it is greater than 1/2 and then becomes complex. The absolute value in the complex plane is also always greater than 1/2. This means that

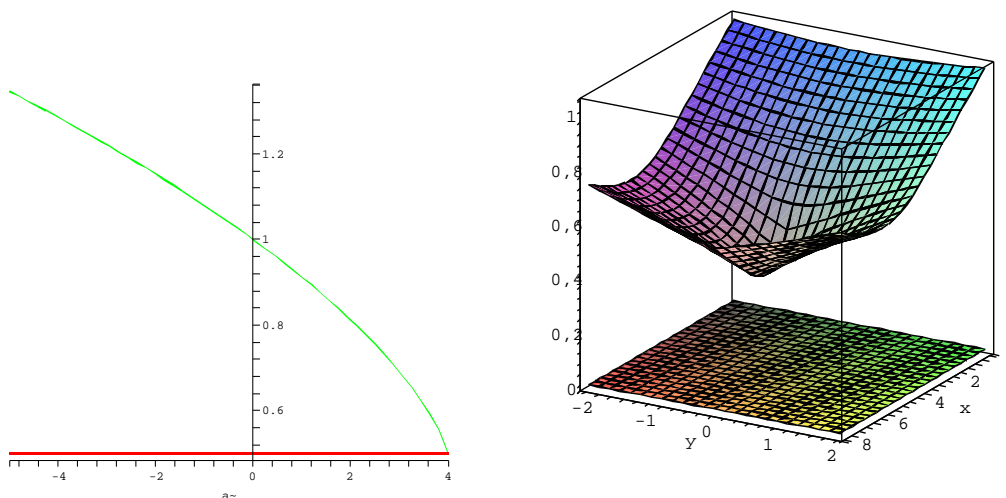


Figure 19: Polarization operator for $D=5$ as a function of p^2/m^2 for real p^2 (left) and the absolute value in the complex plane (right).

depending on the value of dimensionless parameter $\xi = \lambda^2 m$ one has different possibilities: for $\xi < 32\pi^2$ the pole exists at negative real p^2 , for $32\pi^2 < \xi < 64\pi^2$ the pole exists at positive real $p^2 < 4m^2$. For $\xi > 64\pi^2$ there are no poles at all. In this phase a theory is free from unphysical states. A similar situation, but in 4 dimensions, was discussed in [17].

So, it looks like by choosing parameter ξ one can get rid of the unitarity problem. However, it reappears the other way. Indeed, one can see that the denominator of the propagator in this phase becomes negative. It is also negative at $p^2 = 0$. In the scalar case the value of the σ field propagator at $p^2 = 0$ defines the effective potential of ϕ fields after integrating out the auxiliary field σ . This way the negative value of the propagator leads to effective potential with negative quartic coupling, i.e. unbounded from below. In the case of the gauge theory the value of the denominator at $p^2 = 0$ defines the sign of the residue of the gauge field propagator at $p^2 = 0$, i.e. the metric of the gauge field. Negative sign apparently leads to the "wrong" metric which is also not acceptable. Thus, the presence of a pole at the real axis is certainly a problem.

The situation is different in $D = 7$ dimensions. Here one has

$$\Pi_7(p^2) = -\frac{1}{192\pi^3} m^{3/2} \left[\frac{4\sqrt{a}(20 - 3a) - 3(a - 4)^2 \ln\left(\frac{2-\sqrt{a}}{2+\sqrt{a}}\right)}{128\sqrt{a}} \right]. \quad (7.6)$$

Notice the sign difference compared to the $D = 5$ case which means that here there is no pole in the Euclidian region but in the complex plane. In figure 20, we present the same plots as above but for $D=7$.

One can see that the polarization operator in this case can take any value including negative ones. This means that complex conjugate poles exist for any value of $\xi = \lambda m^{3/2}$. As was already mentioned, they correspond to the ghost states and create trouble unless they are canceled.

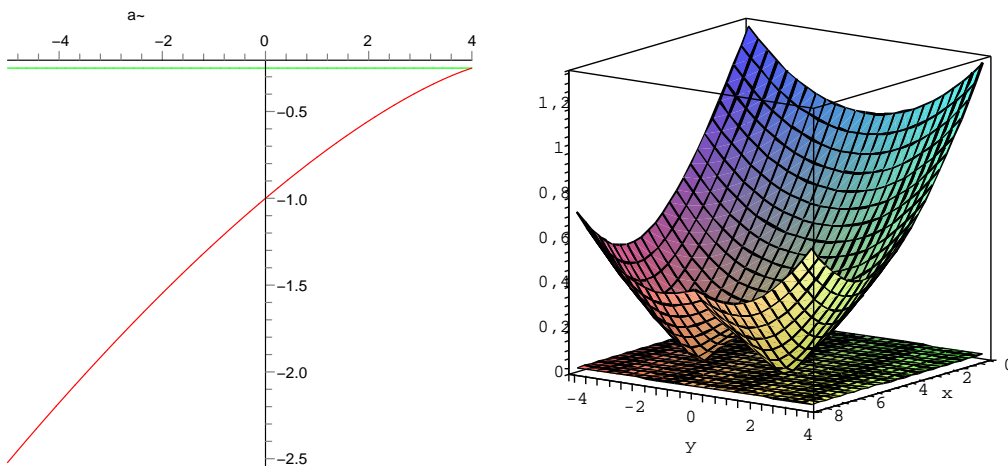


Figure 20: Polarization operator for D=7 as a function of p^2/m^2 for real p^2 (left) and the absolute value in the complex plane (right).

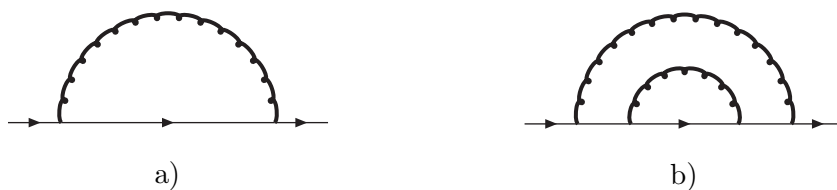


Figure 21: The leading order and next-to-leading order propagator diagrams

We now come to the last step of our analysis. According to ref. [12], in the leading order the contribution of complex conjugated poles to a physical amplitude is canceled, thus preserving the unitarity in physical subspace. To check this, we consider the D=7 case and calculate the contribution from the conjugated ghost poles to the imaginary parts of the Feynman diagrams in the leading and next-to-leading order of the $1/N$ expansion.

Consider first the one loop diagram shown in figure 21a. From the Källén-Lehmann representation for the propagator of the auxiliary field (7.3) we take only the ghost terms ignoring the continuous spectrum. It corresponds to the following integral:

$$\int \frac{dk}{(k-p)^2} \left(\frac{R}{k^2 - M^2} + \frac{R^*}{k^2 - M^{2*}} \right), \tag{7.7}$$

where R and R^* are complex numbers and M^2, M^{2*} are the masses of the conjugated ghost states.

After integration, according to the dimensional regularization prescription, one gets

$$R \int_0^1 dx (1-x)^{3/2} (p^2 x + M^2 + i\Gamma)^{3/2} + R^* \int_0^1 dx (1-x)^{3/2} (p^2 x + M^2 - i\Gamma)^{3/2}. \tag{7.8}$$

For real p^2 the integrand apparently has no imaginary part being the sum of two complex conjugated expressions. The integration does not change this property: the contribution of the ghost states to the imaginary part (to physical amplitude) is canceled and only the continuous spectrum remains. What is crucial here is that the ghost states are conjugated having the opposite sign of the imaginary part and the same real part.

Consider now the next-to-leading order diagram shown in figure 15b. In this diagram there are several ways how the ghosts might enter

1. the inner propagator - non-ghosts, the outer propagator - ghosts;
2. the inner propagator - ghosts, the outer propagator - non-ghosts;
3. the inner propagator - ghosts, the outer propagator - ghosts.

Consider the case when the ghost modes run in the inner propagator and in the outer propagator there is a continuous spectrum (non-ghosts). Then, using the integral representation (7.3) one has the expression

$$\begin{aligned} &\sim \int dm^2 \frac{(m^2)^{3/2}}{\lambda^2(m^2)^3 + (8192\pi^2)^2} \\ &\int \frac{dkdq}{(k^2 - m^2)((p - k)^2)^2(p - k - q)^2} \left(\frac{R}{q^2 - M^2} + \frac{R^*}{q^2 - M^{2*}} \right). \end{aligned} \tag{7.9}$$

Let us first take the integral over q . One has for the ghost part (and similar for the conjugated one)

$$\sim R \int \frac{dk}{(k^2 - m^2)((p - k)^2)^2} \int_0^1 dx ((p - k)^2 x(1 - x) + M^2 x)^{3/2 - \varepsilon}, \tag{7.10}$$

where we keep ε to be finite since the two-loop integral diverges and omit the spectral integration over m^2 . The latter is real and is inessential.

Evaluating the integral over k one gets

$$\begin{aligned} &R \int_0^1 (x(1 - x))^{-1/2 - \varepsilon} (1 - x)^2 dx \int_0^1 dy \int_0^y dz (y - z) z^{-1/2 + \varepsilon} \\ &\left\{ + \Gamma(2\varepsilon) \left(p^2 y(1 - y) + m^2(1 - y) + M^2 \frac{1 - x}{x} z + i\Gamma \frac{1 - x}{x} z \right)^{-2\varepsilon} \right. \\ &[(p^2)^2(1 - y)^4 x^2 + 2p^2(M^2 + i\Gamma)(1 - y)^2 x(M^2 + i\Gamma)^2] \\ &+ \Gamma(-1 + 2\varepsilon) \left(p^2 y(1 - y) + m^2(1 - y) + M^2 \frac{1 - x}{x} z + i\Gamma \frac{1 - x}{x} z \right)^{1 - 2\varepsilon} \\ &[(7 - 2\varepsilon + 2)p^2 x^2(1 - y)^2 + (7 - 2\varepsilon)(M^2 + i\Gamma)x] \\ &+ \Gamma(-2 + 2\varepsilon) \left(p^2 y(1 - y) + m^2(1 - y) + M^2 \frac{1 - x}{x} z + i\Gamma \frac{1 - x}{x} z \right)^{2 - 2\varepsilon} \\ &\left. x^2 \frac{(7 - 2\varepsilon)(3 - 2\varepsilon)}{4} \right\}. \end{aligned}$$

Expanding over ε one has singular and regular parts. The singular part is

$$\begin{aligned} & \frac{R}{2\varepsilon} \int_0^1 (x(1-x))^{-1/2} (1-x)^2 dx \int_0^1 dy \int_0^y dz (y-z) z^{-1/2} \\ & \left\{ + [(p^2)^2(1-y)^4 x^2 + 2p^2(M^2 + i\Gamma)(1-y)^2 x(M^2 + i\Gamma)^2] \right. \\ & - \left(p^2 y(1-y) + m^2(1-y) + M^2 \frac{1-x}{x} z + i\Gamma \frac{1-x}{x} z \right) [9p^2 x^2 (1-y)^2 + 7(M^2 + i\Gamma)x] \\ & \left. + \left(p^2 y(1-y) + m^2(1-y) + M^2 \frac{1-x}{x} z + i\Gamma \frac{1-x}{x} z \right)^2 x^2 \frac{21}{8} \right\}. \end{aligned}$$

The remaining integrals over Feynman parameters are convergent and can be easily evaluated. One can see that due to the presence of $i\Gamma$ the integrand is complex, but adding the complex conjugated term one gets the real polynomial of p^2 .

As for the regular part, it contains

$$\log \left(\left(p^2 y(1-y) + m^2(1-y) + M^2 \frac{1-x}{x} z + i\Gamma \frac{1-x}{x} z \right) \right)$$

and has a cut in momentum plane. However, the logarithm can always be presented in the form $\log(Ae^{i\phi}) = \log(A) + i\phi$, where both the modulus A and the phase ϕ depend on Feynman parameters. This means that adding the conjugated part one again gets the real integrand and, hence, the real function after integration. Here it is again crucial that the ghost states are conjugated and differ only by the sign of the imaginary part.

Thus, we conclude that the contribution from the conjugated ghost states to the imaginary part of the diagram is canceled and, therefore, the ghost states do not contribute to physical amplitudes. The same analysis can be carried out for the other choices of the ghost fields in figure 21b. Moreover, it seems to work in any diagram in all orders of the $1/N$ expansion since the reason for the cancellation is simple and obvious. This means that the unitarity in the physical sector is preserved.

The situation is somewhat similar to that in ref. [31], where the mass generation problem was discussed in the context of higher derivative theory. Besides the physical states there exist non-physical states with a negative norm, but in the asymptotic states the negative norm excitations disappear thus preserving the unitarity of the theory.

8. Conclusion

We conclude that in higher dimensional scalar and gauge theories despite formal non-renormalizability it is possible to construct renormalizable $1/N_f$ expansion which obeys all the rules of a usual perturbation theory. The expansion parameter is dimensionless, the coupling is running logarithmically, all divergencies are absorbed into the renormalization of the wave function and the coupling. The original dimensionful coupling plays a role of a mass and is renormalized multiplicatively. Expansion over this coupling is singular and creates the usual nonrenormalizable terms.

Properties of the $1/N_f$ expansion do not depend on the space-time dimension if it is odd. In even dimensions our formulas after subtraction contain a logarithm which creates some technical problems in calculations but principally do not differ from the odd dimensions.

Since the actual expansion parameter is dimensionless, all the Green functions get logarithmic radiative corrections and the cross-sections decrease with energy like in usual renormalizable theories without violating the unitarity limit. The running of the couplings depends on dimension and does not depend on Abelian or non-Abelian nature of a theory. This may be considered as a drawback of the $1/N_f$ expansion. Unfortunately, the preferable $1/N_c$ expansion cannot be constructed in the same simple manner.

We have demonstrated how one can deal with the problem of unitarity and unphysical pole states. The poles at the real axis can be removed by a proper choice of a dimensionless parameter $\xi = \lambda m^{D/2-2}$ which corresponds to the correct choice of the phase of a theory. However, this does not make a theory reliable. At the same time, the complex conjugated poles remain but fortunately their contribution to the physical amplitudes is canceled. We do not provide a rigorous proof of this cancellation but present the reason for it and several examples how it works in Feynman diagrams. Accepting this reasoning the theory seems to be unitary in physical subspace.

We hope that this approach can be used in extra dimensional theories to get the scattering amplitudes. We expect that the behaviour of the cross-sections will differ from those of the Kaluza-Klein approach [2] being closer to our approach [32] based on the fixed points.

Besides the already mentioned papers [12] there are several attempts to build renormalizable effective quantum gravity using a kind of $1/N$ expansion [33], where the role of an expansion parameter $1/N$ is played by the number of space-time dimensions. The large D limit in this case is very similar to the large N_c planar diagram limit in the Yang-Mills theory considered by 't Hooft [27]. The technique similar to the $1/N_f$ expansion is used also in [34], where the author sums up the soft graviton corrections to the propagator of the scalar field and gets an improved propagator which decreases faster than any power of momenta. Though this partial resummation is similar to the $1/N$ expansion, the absence of an expansion parameter does not justify, to our mind, the selected set of diagrams. From this point of view the $1/N$ expansion is more consistent and contains the guiding line for such a selection.

Acknowledgments

Financial support from RFBR grant # 05-02-17603 and grant of the Ministry of Education and Science of the Russian Federation # 5362.2006.2 is kindly acknowledged. We are grateful to I. Aref'eva, G. Efimov, A. Kotikov, N. Krasnikov, S. Mikhailov, A. Sheplyakov and E. Tomboulis for valuable discussions.

References

- [1] N. Arkani-Hamed, S. Dimopoulos and G.R. Dvali, *The hierarchy problem and new dimensions at a millimeter*, *Phys. Lett. B* **429** (1998) 263 [[hep-ph/9803315](#)];
I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G.R. Dvali, *New dimensions at a millimeter to a Fermi and superstrings at a TeV*, *Phys. Lett. B* **436** (1998) 257 [[hep-ph/9804398](#)];
L. Randall and R. Sundrum, *An alternative to compactification*, *Phys. Rev. Lett.* **83** (1999) 4690 [[hep-th/9906064](#)]; *A large mass hierarchy from a small extra dimension*, *Phys. Rev. Lett.* **83** (1999) 3370 [[hep-ph/9905221](#)].
- [2] Y.A. Kubyshin, *Models with extra dimensions and their phenomenology*, [hep-ph/0111027](#);
J.L. Hewett and M. Spiropulu, *Particle physics probes of extra spacetime dimensions*, *Ann. Rev. Nucl. Part. Sci.* **52** (2002) 397–424 [[hep-ph/0205106](#)];
D.I. Kazakov, *Beyond the standard model*, [hep-ph/0411064](#).
- [3] C. Csáki, C. Grojean, H. Murayama, L. Pilo and J. Terning, *Gauge theories on an interval: unitarity without a Higgs*, *Phys. Rev. D* **69** (2004) 055006 [[hep-ph/0305237](#)].
- [4] D.I. Kazakov and G.S. Vartanov, *Renormalization group treatment of nonrenormalizable interactions*, *J. Phys. A* **39** (2006) 8051 [[hep-th/0509208](#)].
- [5] D.I. Kazakov, *On a generalization of renormalization group equations to quantum field theories of an arbitrary type*, *Theor. Math. Phys.* **75** (1988) 440.
- [6] G.F. Giudice, R. Rattazzi and J.D. Wells, *Quantum gravity and extra dimensions at high-energy colliders*, *Nucl. Phys. B* **544** (1999) 3 [[hep-ph/9811291](#)].
- [7] E. Alvarez and A.F. Faedo, *Renormalized Kaluza-Klein theories*, *JHEP* **05** (2006) 046 [[hep-th/0602150](#)]; *Renormalized masses of heavy Kaluza-Klein states*, *Phys. Rev. D* **74** (2006) 124029 [[hep-th/0606267](#)]; *Quantum corrections to higher-dimensional theories*, [hep-ph/0610424](#).
- [8] R. Rattazzi, *Cargese lectures on extra dimensions*, [hep-ph/0607055](#).
- [9] G. Parisi, *The theory of nonrenormalizable interactions. 1. the large- N expansion*, *Nucl. Phys. B* **100** (1975) 368.
- [10] M. Moshe and J. Zinn-Justin, *Quantum field theory in the large- N limit: a review*, *Phys. Rept.* **385** (2003) 69 [[hep-th/0306133](#)];
J. Zinn-Justin, *Vector models in the large- N limit: a few applications*, [hep-th/9810198](#).
- [11] I.Ya. Aref'eva, *Strong coupling limit for the $O(N)\Phi^4$ interaction*, *Theor. Math. Phys.* **29** (1976) 147; *On the removal of divergences in the model of three-dimensional n field*, *Theor. Math. Phys.* **31** (1977) 3.
- [12] E. Tomboulis, *$1/N$ expansion and renormalization in quantum gravity*, *Phys. Lett. B* **70** (1977) 361; *Renormalizability and asymptotic freedom in quantum gravity*, *Phys. Lett. B* **97** (1980) 77; *Unitarity in higher derivative quantum gravity*, *Phys. Rev. Lett.* **52** (1984) 1173;
I. Antoniadis and E.T. Tomboulis, *Gauge invariance and unitarity in higher derivative quantum gravity*, *Phys. Rev. D* **33** (1986) 2756.
- [13] D.I. Kazakov and G.S. Vartanov, *Renormalizable expansion for nonrenormalizable theories. I: scalar higher dimensional theories*, [hep-th/0607177](#).

- [14] D.I. Kazakov and G.S. Vartanov, *Renormalizable expansion for nonrenormalizable theories. II: gauge higher dimensional theories*, hep-th/0702004.
- [15] G. 't Hooft and M.J.G. Veltman, *Regularization and renormalization of gauge fields*, Nucl. Phys. **B 44** (1972) 189.
- [16] N.V. Krasnikov, *Ultraviolet fixed point behavior of the five-dimensional Yang-Mills theory, the gauge hierarchy problem and a possible new dimension at the TeV scale*, Phys. Lett. **B 273** (1991) 246; *Ultraviolet behavior in five-dimensional Yang-Mills theory*, JETP Lett. **51** (1990) 4;
N.V. Krasnikov and A.B. Kyatkin, *Three-dimensional four fermion (vector) \times (vector) model is renormalizable and ultraviolet finite*, Mod. Phys. Lett. **A 6** (1991) 1315;
N.V. Krasnikov, *The four fermion theory is renormalizable*, Yad. Fiz. **52** (1990) 1516.
- [17] H.J. Schnitzer, *The $1/N$ expansion of renormalizable and nonrenormalizable scalar field theories*, Nucl. Phys. **B 109** (1976) 297.
- [18] V.V. Nesterenko, *On the instability of classical dynamics in theories with higher derivatives*, Phys. Rev. **D 75** (2007) 087703 [hep-th/0612265].
- [19] A.V. Smilga, *Ghost-free higher-derivative theory*, Phys. Lett. **B 632** (2006) 433 [hep-th/0503213];
Benign vs. malicious ghosts in higher-derivative theories, Nucl. Phys. **B 706** (2005) 598 [hep-th/0407231].
- [20] I. Antoniadis, E. Dudas and D.M. Ghilencea, *Living with ghosts and their radiative corrections*, Nucl. Phys. **B 767** (2007) 29 [hep-th/0608094].
- [21] G. 't Hooft, *Dimensional regularization and the renormalization group*, Nucl. Phys. **B 61** (1973) 455.
- [22] M. Beneke, *Renormalons*, Phys. Rept. **317** (1999) 1 [hep-ph/9807443];
M. Beneke and V. Braun, *Renormalons and power corrections in B.Ioffe Festschrift*, World Scientific, Singapore (2001).
- [23] D. Anselmi, *Large- N expansion, conformal field theory and renormalization-group flows in three dimensions*, JHEP **06** (2000) 042 [hep-th/0005261].
- [24] J.A. Gracey, *Algorithm for computing the beta function of quantum electrodynamics in the large- N_f expansion*, Int. J. Mod. Phys. **A 8** (1993) 2465 [hep-th/9301123]; *Electron mass anomalous dimension at $O(1/(N_f^2))$ in quantum electrodynamics*, Phys. Lett. **B 317** (1993) 415 [hep-th/9309092].
- [25] J.A. Gracey, *Quark, gluon and ghost anomalous dimensions at $O(1/N_f)$ in quantum chromodynamics*, Phys. Lett. **B 318** (1993) 177 [hep-th/9310063]; *The QCD beta-function at $O(1/n_f)$* , Phys. Lett. **B 373** (1996) 178 [hep-ph/9602214].
- [26] K.R. Dienes, E. Dudas and T. Gherghetta, *Extra spacetime dimensions and unification*, Phys. Lett. **B 436** (1998) 55 [hep-ph/9803466]; *Grand unification at intermediate mass scales through extra dimensions*, Nucl. Phys. **B 537** (1999) 47 [hep-ph/9806292].
- [27] G. 't Hooft, *A planar diagram theory for strong interactions*, Nucl. Phys. **B 72** (1974) 461.
- [28] G. Källén, *Helv. Phys. Acta* **25** (1952) 417; *Quantum electrodynamics*, Springer-Verlag, Berlin (1972);
H. Lehmann, *On the Properties of propagation functions and renormalization constants of quantized fields*, Nuovo Cim. **11** (1954) 342.

- [29] R.E. Cutkosky, *Singularities and discontinuities of Feynman amplitudes*, *J. Math. Phys.* **1** (1960) 429.
- [30] I.Ya.Aref'eva, *Phase transition in the three-dimensional chiral field*, *Ann. Phys. (NY)* **117** (1979) 393.
- [31] A.A. Slavnov, *Higgs mechanism as a collective effect due to extra dimension*, *Theor. Math. Phys.* **148** (2006) 1159 [[hep-th/0604052](#)]; *Hierarchy of massive gauge fields*, *Phys. Lett.* **B 620** (2005) 97 [[hep-th/0505195](#)]; *Renormalizable electroweak model without fundamental scalar mesons*, [hep-th/0601125](#).
- [32] D.I. Kazakov and G.S. Vartanov, *On high energy scattering in extra dimensions*, *Theor. Math. Phys.* **147** (2006) 533 [[hep-ph/0410342](#)].
- [33] F. Canfora, *The w behavior of gravity at large- N* , *Phys. Rev.* **D 74** (2006) 064020 [[hep-th/0608203](#)]; *A large- N expansion for gravity*, *Nucl. Phys.* **B 731** (2005) 389 [[hep-th/0511017](#)];
N.E.J. Bjerrum-Bohr, *Quantum gravity at a large number of dimensions*, *Nucl. Phys.* **B 684** (2004) 209 [[hep-th/0310263](#)];
A. Strominger, *The inverse dimensional expansion in quantum gravity*, *Phys. Rev.* **D 24** (1981) 3082.
- [34] B.F.L. Ward, *Exact quantum loop results in the theory of general relativity*, [hep-ph/0607198](#); *Quantum corrections to newton's law*, *Mod. Phys. Lett.* **A 17** (2002) 2371 [[hep-ph/0204102](#)]; *Are massive elementary particles black holes?*, *Mod. Phys. Lett.* **A 19** (2004) 143 [[hep-ph/0305058](#)];
Massive elementary particles and black holes, *JCAP* **02** (2004) 011 [[hep-ph/0312188](#)].